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Subharmonic Oscillations of Forced Pendulum-Type Equations

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1. INTRODUCTION

In this paper we are concerned with the existence of subharmonic solutions of second order differential equations of the form

$$\ddot{x} + g(x) = f(t),$$

where f is periodic with minimal period T and mean value zero. We have in mind as a particular case the pendulum equation, where $g(x) = A \sin x$.

First results on the existence of subharmonic orbits in a neighborhood of a given periodic motion were obtained by Birkhoff and Lewis (cf. [3] and [14]) by perturbation-type techniques. Rabinowitz [15] was able to prove the existence of subharmonic solutions for Hamiltonian systems by the use of variational methods. His approach is not of local type like the one in [3], and enables one to obtain a sequence of solutions whose minimal period tends toward infinity in the case when the Hamiltonian function has subquadratic or superquadratic growth. These results have been extended in various directions, cf. [2, 5, 6, 8, 13, 16–18]. Local results on subharmonics for the forced pendulum equation can be found in [19].

Hamiltonian systems with periodic nonlinearity were studied by Conley and Zehnder [6]. They proved the existence of subharmonic solutions under some assumptions on the nondegenerateness of the solutions, by the use of Morse–Conley theory.

In this paper we will prove the existence of subharmonic oscillations of a pendulum-type equation by the use of classical Morse theory together with an iteration formula for the index due to Bott [4] and developed in [7] and [1].

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2. The Main Result

Let T be a fixed positive number and $k \geqslant 2$ an integer. Assume $f: \mathbb{R} \to \mathbb{R}$ to be a continuous periodic function, with minimal period T, and such that

$$\int_0^T f(t)\,dt = 0. \tag{1}$$

We consider the equation

$$\ddot{x}(t) + g(x(t)) = f(t), \tag{2}$$

where $g: \mathbb{R} \to \mathbb{R}$ is a continuous function such that, setting

$$G(x) = \int_0^x g(s)\,ds,$$

the function G is 2π-periodic.

We want to prove the existence of subharmonic solutions; i.e., we look for periodic solutions of (2) having kT as minimal period. The kT-periodic solutions of (2) correspond to the critical points of the functional ϕ_k, defined on the Hilbert space $H^1_{kT} = \{x \in H^1([0, kT]) : x(0) = x(kT)\}$ as follows:

$$\phi_k(x) = \int_0^{kT} \left[\tfrac{1}{2}(\dot{x}(t))^2 - G(x(t)) + f(t)\,x(t)\right] dt. \tag{3}$$

However, the critical points of ϕ_k do not necessarily correspond to periodic solutions of (2) with *minimal* period kT, as can be seen from the case $g \equiv 0$. In fact, in this case the kT-periodic solutions of (2) are of the form

$$x(t) = C_0 - t\,\frac{1}{kT}\int_0^{kT}\left(\int_0^s f(u)\,du\right) ds + \int_0^t \left(\int_0^s f(u)\,du\right) ds, \tag{4}$$

where $C_0 = x(0)$ can be chosen arbitrarily in $\mathbb{R}$. Because of (1),

$$\frac{1}{kT}\int_0^{kT}\left(\int_0^s f(u)\,du\right) ds = \frac{1}{T}\int_0^{T}\left(\int_0^s f(u)\,du\right) ds,$$

and then any $x(t)$, as in (4), has in fact period T.

It can be shown, cf. [10–12], that the functional ϕ_k is bounded from below and satisfies the Palais–Smale condition. So ϕ_k always has a minimum. If $g \equiv 0$, the minimum points of ϕ_k are as in (4), where C_0 is an arbitrary real number. In particular, they are not isolated.

Let x_0 be a T-periodic solution of Eq. (2). Define, for λ and t in $\mathbb{R}$, the matrix

$$A_\lambda(t) = \begin{pmatrix} 0 & -1 \\ \lambda + g'(x_0(t)) & 0 \end{pmatrix}$$

and consider the fundamental solution $X_\lambda(t)$ which satisfies

$$\dot{X}_\lambda(t) = A_\lambda(t)\, X_\lambda(t)$$
$$X_\lambda(0) = Id.$$

It is well known (see e.g. [9]) that the eigenvalues $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ of $X_\lambda(T)$ have the following properties:

(i) either both $\sigma'_{\lambda,T}$ and $\sigma''_{\lambda,T}$ are in $\mathbb{R}$, or $\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T}$;

(ii) $\sigma'_{\lambda,T} \cdot \sigma''_{\lambda,T} = 1$;

(iii) there exists $\lambda_0 < \lambda_1$ such that the maps $\lambda \mapsto \sigma'_{\lambda,T}$ and $\lambda \mapsto \sigma''_{\lambda,T}$ are continuous and one to one if $\lambda_0 \leqslant \lambda \leqslant \lambda_1$. Moreover,

$$0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T} \qquad (\lambda < \lambda_0),$$
$$\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T} \in S^1 \qquad (\lambda_0 \leqslant \lambda \leqslant \lambda_1).$$

The T-periodic solution x_0 is said to be nondegenerate if $1 \notin \{\sigma'_{0,T}, \sigma''_{0,T}\}$.

Given $\sigma \in S^1$, we define $J(x_0, T, \sigma)$ to be the number of negative λ's for which $\sigma \in \{\sigma'_{\lambda,T}, \sigma''_{\lambda,T}\}$. The number $J(x_0, T, 1)$ is then the Morse index of the T-periodic solution x_0.

We are now able to formulate our main result.

THEOREM 1. *Assume the following conditions*:

(a) *the T-periodic solutions of Eq.* (2) *are isolated*;

(b) *every T-periodic solution of* (2) *having Morse index equal to zero is nondegenerate.*

Then there exists a $k_0 \geqslant 2$ *such that, for every prime integer* $k \geqslant k_0$, *there is a periodic solution of* (2) *with minimal period* kT.

Remarks. (1) We have seen above that in the case $g \equiv 0$ there are no subharmonic solutions of (2), and the T-periodic solutions are not isolated, and therefore degenerate. So neither (a) nor (b) is verified in this case.

(2) In [6], Conley and Zehnder proved the existence of subharmonic solutions for a system with Hamiltonian function periodic in each of its variables. They showed that when all the T-periodic solutions, together with their iterates, are nondegenerate, then there exists a periodic solution with minimal period kT if k is a sufficiently large prime number.

We do not need to assume, as in [6], that also the iterates of the T-periodic solutions of (2) are nondegenerate. Since for a T-periodic solution x_0 one has $\sigma'_{\lambda,kT}=(\sigma'_{\lambda,T})^k$ and $\sigma''_{\lambda,kT}=(\sigma''_{\lambda,T})^k$, it could then happen in principle that $1\in\{\sigma'_{\lambda,kT},\sigma''_{\lambda,kT}\}$ even if $1\notin\{\sigma'_{\lambda,T},\sigma''_{\lambda,T}\}$.

Proof of Theorem 1. Let us introduce the Hilbert space

$$\tilde{H}_k=\left\{\tilde{x}\in H^1_{kT}:\int_0^{kT}\tilde{x}(t)\,dt=0\right\}.$$

By (1) and the 2π-periodicity of G, we have that

$$\phi_k(x+2\pi)=\phi_k(x)$$

for every $x\in H^1_{kT}$. Set $S^1=\mathbb{R}/(2\pi\mathbb{Z})$. It is then equivalent to consider the functional ψ_k defined on $S^1\times\tilde{H}_k$ by

$$\psi_k(x)=\phi_k(\bar{x}+\tilde{x})$$

for every $x=(\bar{x},\tilde{x})\in S^1\times\tilde{H}_k$. The functionals ψ_k are bounded from below and satisfy the Palais–Smale condition (cf. [10–12]). By assumption (a), the functional ψ_1 has only a finite number of critical points $x_0, ..., x_n$. It is clear that the functions x_i $(0\leqslant i\leqslant n)$, extended by T-periodicity on $[0, kT]$, are also critical points of ψ_k for $k\geqslant 2$.

We now assert the following.

Claim. There exists an integer k_0 such that, for $k\geqslant k_0$ and $0\leqslant i\leqslant n$, either $J(x_i, kT, 1)=0$ and x_i is nondegenerate, or $J(x_i, kT, 1)\geqslant 2$.

Assume for the moment that the above Claim holds true. In case $k\geqslant k_0$ is a prime number, since f has minimal period T, the critical points of ψ_k have as minimal period either T or kT. Assume by contradiction that $x_0, ..., x_n$ are the oly critical points of ψ_k. Since the Poincaré polynomial of $S^1\times\tilde{H}_k$ is $(1+t)$, we have

$$\sum_{i=0}^{n} P_k(t, x_i)=(1+t)[1+Q(t)], \tag{5}$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients and $P_k(t,x_i)=\sum_j \dim C_j(\psi_k, x_i)t^j$ is the usual Morse polynomial of x_i (see e.g. [12]). By the Claim, if $J(x_i, kT, 1)=0$, then $P_k(t, x_i)=1$. Otherwise, if $J(x_i, kT, 1)\geqslant 2$, then $\dim C_j(\psi_k, x_i)=0$ for $j=0, 1$. This implies that Eq. (5) can never be satisfied, and we have a contradiction.

To conclude the proof of the theorem we need then to prove the above Claim. In order to do so, let x_i be a critical point of ψ_1 and let $\lambda_0<\lambda_1$ be as in property (iii). First of all, we claim that $\lambda_0\neq 0$. Indeed, if on the contrary $\lambda_0 = 0$, we would have, for every negative λ, $0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T}$, which implies $J(x_i, T, 1) = 0$. On the other hand, by (iii), $\sigma'_{0,T} = 1 = \sigma''_{0,T}$, so that x_i would be a degenerate T-periodic solution with Morse index equal to zero, in contradiction with assumption (b).

Suppose $\lambda_0 > 0$. Then, for every $\lambda \leqslant 0$, we have $0 < \sigma'_{\lambda,T} < 1 < \sigma''_{\lambda,T}$ and hence $J(x_i, T, \sigma) = 0$ for every $\sigma \in S^1$. By [4, Theorem 1] we have

$$J(x_i, kT, 1) = \sum_{\sigma^k = 1} J(x_i, T, \sigma) = 0.$$

Moreover x_i, as a critical point of ψ_k, is also nondegenerate, since

$$0 < \sigma'_{0,kT} = (\sigma'_{0,T})^k < 1 < \sigma''_{0,kT} = (\sigma''_{0,T})^k.$$

Suppose now $\lambda_0 < 0$. Then for every $\lambda \in]\lambda_0, \lambda_0 + \varepsilon[$, for $\varepsilon > 0$ small enough, we have $\sigma'_{\lambda,T} = \bar{\sigma}''_{\lambda,T} \in S^1$ and

$$J(x_i, T, \sigma'_{\lambda,T}) = J(x_i, T, \sigma''_{\lambda,T}) > 0.$$

Hence, for k large enough, we have

$$J(x_i, kT, 1) = \sum_{\sigma^k = 1} J(x_i, T, \sigma) \geqslant 2.$$

This proves the Claim, and completes the proof of Theorem 1.

Under a stronger assumption, in the following theorem we will obtain the existence of *two* subharmonic oscillations.

THEOREM 2. *Suppose that the kT-periodic solutions of* (2) *are nondegenerate for $k = 1$ and for every prime integer k. Then there exists $k_0 \geqslant 3$ such that, for every prime integer $k \geqslant k_0$, there are two geometrically distinct periodic solutions of* (2) *with minimal period kT.*

Proof. As a consequence of the assumption, for every prime number k, the number n_k of critical points of ψ_k is finite. Since the Poincaré polynomial of $S^1 \times \tilde{H}_k$ is $(1+t)$, n_k must be even. It follows from Theorem 1 that, for $k \geqslant k_0$, $n_k \geqslant n_1 + k$. Then $n_k \geqslant n_1 + k$, and the proof is complete.

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